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RE-IONIZATION AND ITS IMPRINT ON THE COSMIC MICROWAVE BACKGROUND

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ABSTRACT

Early reionization changes the pattern of anisotropies expected in the cosmic microwave background. To explore these changes, we derive from first principles the equations governing anisotropies, focusing on the interactions of photons with electrons. Vishniac (1987) claimed that second order terms can be large in a re-ionized Universe, so we derive equations correct to second order in the perturbations. There are many more second order terms than were considered by Vishniac. To understand the basic physics involved, we present a simple analytic approximation to the first order equation. Then turning to the second order equation, we show that the Vishniac term is indeed the only important one. We also present numerical results for a variety of ionization histories [in a standard cold dark matter Universe] and show quantitatively how the signal in several experiments depends on the ionization history. The most pronounced indication of a re-ionized Universe would be seen in very small scale experiments; the expected signal in the Owens Valley experiment is smaller by a factor of order ten if the last scattering surface is at a redshift $z \simeq 100$ as it would be if the Universe were re-ionized very early. On slightly larger scales, the expected signal in a re-ionized Universe *is* smaller than it would be with standard recombination, but only by a factor of two or so. The signal is even smaller in these experiments in the intermediate case where *some* photons last scattered at the standard recombination epoch.

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1. Introduction

Measurements of the cosmic microwave background (CMB) are very powerful probes of theories of structure formation. Even the so-called "medium-angle" CMB experiments are looking at scales larger than those in the largest redshift surveys. This means that CMB observations are some of the purest observations in the field: more than any other type of experiment they sample the primordial spectrum of fluctuations. We can therefore hope to use the CMB experiments to answer many questions about primordial fluctuations: Are they Gaussian? What is the spectral index? Are they adiabatic or isocurvature? Do gravity waves play any role?

These are profound questions and the possibility that we might answer them is one of the great promises of observational cosmology. However, there are several flies in the ointment. Perhaps most disturbing is foreground: our galaxy has lots of dust, synchrotron and free-free emission. All of these can get in the way of detecting the purely cosmic radiation. Another potential barrier to inferring the primordial spectrum from CMB measurements – and the one we wish to focus on here – is the possibility of *reionization.* In the standard cosmology the electrons and protons "recombine" at redshift $z \simeq 1100$, thereby cutting off contact between photons and matter. However, it is possible that hydrogen was "re-ionized" at a later epoch. The subsequent contact between photons and free electrons changes the pattern of anisotropies in the CMB. Thus *we can use the CMB to infer the ionization history of our Universe.* As we have said, knowing this history is crucial for the sake of determining the primordial spectrum, but it is also interesting in of itself. For example, if we determine that the Universe was ionized at a redshift $z \sim 100$, we will have learned something very useful about structure at that epoch.

In this paper, we would like to quantify these statements with particular attention to experiments. The simplest way to discuss anisotropies in the CMB is in $l-$ space (Bond *et al.* 1991). That is, we expand $\delta T(\theta,\phi)/T = \sum_{lm} a_{lm} Y_{lm}(\theta,\phi)$ and define $C_l \equiv \langle |a_{lm}|^2 \rangle$, where the angular brackets denote ensemble averages. Each theory has its own set of predicted C_l's and therefore predicts that a given experiment will observe a variance

$$\langle (\Delta T/T)^2_{\rm expt} \rangle = \sum_{l=2}^{\infty} \frac{2l+1}{4\pi} C_l W_{l,{\rm expt}} \tag{1.1}$$

where W_l is the window function appropriate to that experiment.

Figure 1 shows the window functions of a variety of experiments. For large l, the expected variance in a given experiment is seen from Eq. (1.1) to be roughly $1/2\pi \int d\ln(l)\ (l^2 C_l) W_l$. So the quantity $(l^2 C_l)$ is convolved with the window function to give the expected variance. Figure 2 plots the $l^2 C_l$ predicted by cold dark matter (CDM) with standard ionization history. The peak at $l = 200$ is probed by several of the experiments shown in Figure 1.

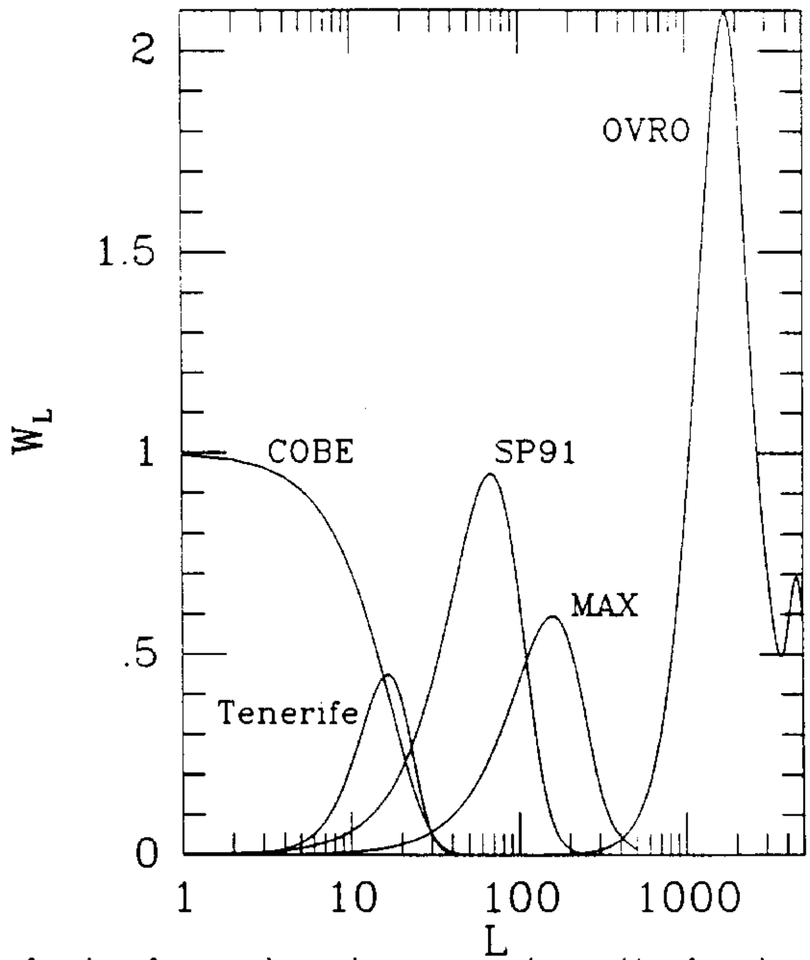

FIG. 1. Window functions for several experiments currently searching for anisotropies in the cosmic microwave background. The curve labeled SP91 denotes the filter for the ACME telescope for the nine-point scan of 1991 (Gaier et al. 1992) in the South Pole; TENERIFE the one used by Davies et al. (1987) at Tenerife; MAX the filter used by Meinhold et al. (1993) in the region of μ-Pegasus; OVRO the filter used at Owens Valley (Readhead et al. 1989). For reference the COBE (Smoot et al. 1992) filter is also plotted.

We want to know how the C_l's change if the Universe is re-ionized. Is there still a peak at $l = 200$? Does the amplitude change? Does the peak move, perhaps to lower or higher l? Or both? To answer these questions we derive the fundamental Boltzmann equation governing the interaction between photons and electrons, the interaction that, as we will see, is responsible for the peak in Fig. 2. This derivation is presented in sections in what we think is a very systematic way. One of the advantages of this systematic treatment is it will enable us to pick up not only the linear terms [which of course have already been derived many times (Peebles & Yu 1970; Wilson & Silk 1981)] but also the second order terms. These are of particular interest because Vishniac (1987) has analyzed one such second order term and concluded that in a reionized Universe, anisotropies are *generated* at small scales [high l]. By introducing this systematic treatment of the Boltzmann equation, we

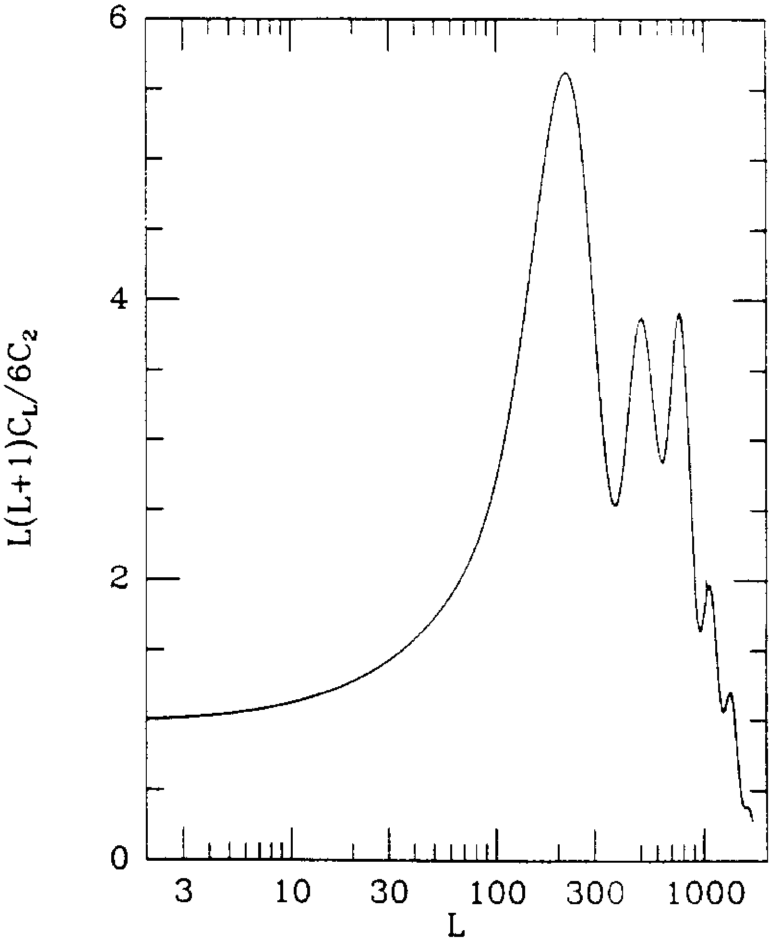

FIG. 2. The moments predicted by CDM with standard recombination. The quantity plotted $l(l+1)C_l/6C_2$ is equal to one at low l where only the Sachs-Wolfe effect is significant. In the post-COBE era, we know that $l(l+1)C_l/6C_2 = l(l+1)C_l/(2.4 \times 10^{-5})^2$.

will be able to see if there are other second order terms that are as large as those considered by Vishniac. and if cancellations occur.

First though, in section 4, we solve the first order equations. The focus is on two possible ionization histories: the standard scenario wherein electrons and protons recombine at $z \sim 1100$ with no further ionization and the opposite extreme wherein electrons remain free throughout the whole history of the Universe. In addition to presenting the results of a full numerical treatment, we also solve the equation [or an approximation to it] analytically to gain insight into the location of the peak in $l-$ space. We find that this peak shifts to *lower* l if the Universe is re-ionized early enough. If the Universe never recombines. then there is a peak at $l \sim 50$. The signal in an experiment with a filter centered around this value of l would see a *larger* signal in a completely re-ionized Universe than in one with standard recombination. We also show why $l^2 C_l$ falls off at high l. This will lead into a discussion of the second order terms expected to be important. Sections 5 and 6 use this information to complete the derivation of the second order Boltzmann equation and present the solution.

Our techniques are similar to those of Efstathiou (1988). In particular we will recapture Vishniac's result and show that the term he identified is indeed the dominant one. Thus *we support Vishniac's conclusions about second order effects.* Finally in section 7. we get more specific about experiments, presenting the predicted $\langle(\Delta T/T)^2_{\rm expt}\rangle$ in a variety of experiments for a variety of ionization histories.

One final introductory comment: The signal from the cosmic background depends on many parameters even in a model as simple as CDM. For example, the spectral index n need not be equal to one, and even a slight deviation has dramatic implications for the small and medium scale measurements. The signal in a given experiment also depends (Bond *et al.* 1991; Dodelson & Jubas 1993) on Ω_B, the fraction of critical density in baryons today, and h which parametrizes the Hubble constant today ($H_0 = 100h\text{kmsec}^{-1}\text{Mpc}^{-1}$). And of course, models other than CDM give different predictions and have different sets of parameters to fiddle with. Here we are focusing only on the impact of different ionization histories. It therefore makes sense to fix all these other parameters and play with the one variable of interest: the ionization history. Accordingly, we will focus only on a cold dark matter dominated Universe with Harrison-Zel'dovich spectrum ($n = 1$) and set $h = 1/2$ and $\Omega_B = 0.05$. Several other groups (Bond & Efstathiou 1987; Efstathiou 1988; Hu, Scott, & Silk 1993) have recently considered the effects of re-ionization for other cosmologies, in particular the minimal isocurvature model proposed by Peebles (1987).

2. Compton Collision Term: A General Derivation

The photon spectrum is governed by the Boltzmann equation:

$$\frac{d}{dt}f(\mathbf{x},\mathbf{p},t) = C(\mathbf{x},\mathbf{p},t). \tag{2.1}$$

Here f is the photon occupation number, a function of momentum $\mathbf{p}$, position $\mathbf{x}$, and time t. The collision term, C, also depends on these variables. Theoretically, it includes contributions from all scattering processes, although in practice only Compton scattering off free electrons need be considered.

In the absence of collisions [$C = 0$], Eq. (2.1) says simply that photons travel freely along geodesics. For example, in a Robertson- Walker background, the left hand side of Eq. (2.1) becomes

$$\frac{df}{dt} = \left\{\frac{\partial}{\partial t} + \frac{\mathbf{p}}{p}\cdot\frac{\partial}{\partial \mathbf{x}} - Hp\frac{\partial}{\partial p}\right\} f(\mathbf{x},\mathbf{p},t) \tag{2.2}$$

where H is the Hubble rate and $p = |\mathbf{p}|$. If we are interested in large scale anisotropies, we must do better: we must account for the perturbed metric when expanding d/dt in terms of partial derivatives. Such an account leads to the Sachs-Wolfe effect (Sachs & Wolfe 1967). [Recently, Martinez-Gonzalez, *etal.* (1992) have taken this program a step further and considered second order effects in the perturbations to the metric.]

In this paper we will focus on the right hand side of Eq. (2.1): the collision term. This term governs small scale anisotropies and spectral distortions. In the limit of completely elastic collisions, the right hand side vanishes. Typically, in the regime of interest, very little energy is imparted from electrons to photons in a collision, so this limit is a good approximation. To get non-zero effects, therefore, we simply need to expand the right hand side systematically in powers of the energy transfer.

The starting point then is the collision term corresponding to the process

$$e(\mathbf{q})\gamma(\mathbf{p}) \leftrightarrow e(\mathbf{q}')\gamma(\mathbf{p}').$$

To calculate this, we find the matrix element M for the process, square it, weight it by the occupation numbers of the particles and integrate over all other momenta, $\mathbf{q}, \mathbf{q}', \mathbf{p}'$. Therefore,

$$C(p) = \frac{1}{p}\int \frac{d^3\mathbf{q}}{(2\pi)^3 2E(\mathbf{q})}\frac{d^3\mathbf{q}'}{(2\pi)^3 2E(\mathbf{q}')}\frac{d^3\mathbf{p}'}{(2\pi)^3 2E(\mathbf{p}')}(2\pi)^4\delta^4(q+p-q'-p')|M|^2$$
$$\times\left[g(\mathbf{q}')f(\mathbf{p}')\Big(1+f(\mathbf{p})\Big) \quad - \quad g(\mathbf{q})f(\mathbf{p})\Big(1+f(\mathbf{p}')\Big)\right] \tag{2.3}$$

where $E(q) = \sqrt{q^2+m^2}$ and the delta function enforces energy-momentum conservation. The last line in Eq. (2.3) contains the distribution functions, f that of the photons and g the electrons. We have dropped the Pauli suppression factors $1-g$, since in all realistic cosmological scenarios, g is very small. We don't know f [that is what we are trying to solve for] but we do know g: Due to the fast rate of Coulomb collisions, the electrons are kept in thermal equilibrium, so

$$g(\mathbf{q}) = n_e\left(\frac{2\pi}{m_e T_e}\right)^{3/2}\exp\left\{\frac{-(\mathbf{q}-m_e\mathbf{v})^2}{2m_e T_e}\right\}. \tag{2.4}$$

where $\mathbf{v}$ is the velocity of the electrons and the normalization comes from requiring that $\int d^3q\, g/(2\pi)^3 = n_e$, the electron density. Before beginning the expansion, we can trivially do the $\mathbf{q}'$ integration in Eq. (2.3) by using the three dimensional delta function. This leaves

$$C(p) = \frac{1}{8\pi p}\int dp'p'\frac{d\Omega'}{4\pi}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{|M|^2}{E(\mathbf{q})E(\mathbf{p}+\mathbf{q}-\mathbf{p}')}\delta\Big(p+E(\mathbf{q})-p'-E(\mathbf{p}+\mathbf{q}-\mathbf{p}')\Big)$$
$$\times\left[g(\mathbf{p}+\mathbf{q}-\mathbf{p}')f(\mathbf{p}')\Big(1+f(\mathbf{p})\Big) \quad - \quad g(\mathbf{q})f(\mathbf{p})\Big(1+f(\mathbf{p}')\Big)\right] \tag{2.5}$$

As mentioned above, the energy transfer, $E(\mathbf{q}) - E(\mathbf{q}+\mathbf{p}-\mathbf{p}')$, is small compared with the typical thermal energies which are of order T. In fact, the energy difference is of order $E(\mathbf{q}) - E(\mathbf{q}+\mathbf{p}-\mathbf{p}') \simeq (\mathbf{p}'-\mathbf{p})\cdot\mathbf{q}/m_e = \mathcal{O}(Tq/m_e)$. Thus our expansion parameter, the energy difference over the temperature, is actually q/m_e. The electron momentum has two sources: the bulk velocity ($q = m_e v$) and the thermal motion ($q \sim \sqrt{m_e T}$). Thus, an expansion in q/m_e is necessarily an expansion in v and $\sqrt{T/m_e}$. At the end of our expansion we will have a first order source term linear in v and a host of second order terms quadratic in v and $\sqrt{T/m_e}$.

The strategy now is to expand everything – energies, squared matrix element, delta function and distribution functions – using the energy transfer as an expansion parameter. The Boltzmann distribution expands to

$$g(\mathbf{p}+\mathbf{q}-\mathbf{p}') = g(\mathbf{q})\Bigg\{1 - \frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{q}-m\mathbf{v})}{m_e T_e} - \frac{(\mathbf{p}-\mathbf{p}')^2}{2m_e T_e} + \frac{1}{2}\left[\frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{q}-m\mathbf{v})}{m_e T_e}\right]^2 + \ldots\Bigg\} \tag{2.6}$$

The second term in brackets is first order in the perturbative quantities while the last two are second order. Meanwhile the delta function expands to

$$\delta\Big(p + E(\mathbf{q}) - p' - E(\mathbf{p}+\mathbf{q}-\mathbf{p}')\Big) = \delta(p-p') + \frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'} + \frac{(\mathbf{p}-\mathbf{p}')^2}{2m_e}\frac{\partial\delta(p-p')}{\partial p'} + \frac{1}{2}\left[\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{q}}{m_e}\right]^2\frac{\partial^2\delta(p-p')}{\partial p'^2} + \ldots \tag{2.7}$$

The derivatives of the Dirac delta functions here will ultimately be handled by integration by parts. [Bernstein (1988) introduced this approach to derive the Kompaneets equation which, as will see in section 5, is a special case of the general second order equation.] Finally, we expand the photon distribution function,

$$f = f^{(0)}(p) + f^{(1)}(\mathbf{p}) + f^{(2)}(\mathbf{p}) \tag{2.8}$$

where $f^{(0)}(p)$ is the zero order photon distribution [typically Planckian] which of course depends only on the magnitude of p. We can use these three expansions to rewrite Eq. (2.5) as

$$\begin{aligned} C(p) = \frac{1}{8\pi p}\int dp'p'\frac{d\Omega'}{4\pi}\int\frac{d^3\mathbf{q}}{(2\pi)^3}\frac{|M|^2 g(\mathbf{q})}{E(\mathbf{q})E(\mathbf{p}+\mathbf{q}-\mathbf{p}')}\Bigg\{\delta(p-p') + \frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'} \\ + \frac{(\mathbf{p}-\mathbf{p}')^2}{2m_e}\frac{\partial\delta(p-p')}{\partial p'} + \frac{1}{2}\left[\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{q}}{m_e}\right]^2\frac{\partial^2\delta(p-p')}{\partial p'^2}\Bigg\} \\ \times\Bigg\{\Big[f^{(0)}(p') - f^{(0)}(p)\Big] + \Big[f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p}) - f^{(0)}(p')(1+f^{(0)}(p))\frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{q}-m_e\mathbf{v})}{m_e T_e}\Big] \\ + \Big[f^{(2)}(\mathbf{p}') - f^{(2)}(\mathbf{p}) - \frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{q}-m_e\mathbf{v})}{m_e T_e}\Big(f^{(1)}(\mathbf{p}')(1+f^{(0)}(p)) + f^{(0)}(p')f^{(1)}(\mathbf{p})\Big) \\ + f^{(0)}(p')(1+f^{(0)}(p))\left(\frac{-(\mathbf{p}-\mathbf{p}')^2}{2m_e T_e} + \frac{1}{2}\left(\frac{(\mathbf{p}-\mathbf{p}')\cdot(\mathbf{q}-m_e\mathbf{v})}{m_e T_e}\right)^2\right)\Big]\Bigg\}. \end{aligned} \tag{2.9}$$

This equation looks like a mess, and we have not even expanded all the quantities on the first line yet. It turns out though that the hard part is over. We now recognize that the zero order term in Eq. (2.9), i.e. the one we get when multiplying together the first terms in each of the curly brackets, vanishes. That is, $\delta(p-p')(f^{(0)}(p)-f^{(0)}(p'))$ is zero. Therefore only terms of first order remain after multiplying all the terms in the two curly brackets. This means that [since we are interested only in terms up to second order] we only have to keep first order terms when we expand the matrix element and the energies on the first line. In fact to first order the energies can be simply replaced by m_e, and the matrix element squared is

$$|M|^2 = 6\pi\sigma_T m_e^2 \left((1+\cos^2\theta) - 2\cos\theta(1-\cos\theta)\mathbf{q}\cdot(\hat{\mathbf{p}}+\hat{\mathbf{p}}')/m_e + \ldots\right) \tag{2.10}$$

where $\cos\theta = \hat{\mathbf{p}}\cdot\hat{\mathbf{p}'}$ and σ_T is the Thomson cross section. To simplify things further, we can explicitly do the d^3q integral by using $\langle g\rangle = n_e$; $\langle g\mathbf{q}\rangle = n_e m_e \mathbf{v}$; and $\langle g q_i q_j\rangle = \delta_{ij} n_e m_e T_e + n_e m_e^2 v_i v_j$, where $\langle A\rangle = \int d^3q\ A/(2\pi)^3$. Then we find

$$\begin{aligned} C(\mathbf{p}) = \frac{3n_e\sigma_T}{4p}\int dp'p'\frac{d\Omega'}{4\pi}\Big[&c^{(1)}(\mathbf{p},\mathbf{p}') + c_K^{(2)}(\mathbf{p},\mathbf{p}') + c_\Delta^{(2)}(\mathbf{p},\mathbf{p}') \\ &+ c_{\Delta v}^{(2)}(\mathbf{p},\mathbf{p}') + c_{vv}^{(2)}(\mathbf{p},\mathbf{p}')\Big] \end{aligned} \tag{2.11}$$

where the first order integrand is

$$\begin{aligned} c^{(1)}(\mathbf{p},\mathbf{p}') = (1+\cos^2\theta)\Bigg(&\delta(p-p')(f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p})) \\ &+ (f^{(0)}(p')-f^{(0)}(p))(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}\frac{\partial\delta(p-p')}{\partial p'}\Bigg); \end{aligned} \tag{2.12}$$

and we have separated the second order terms into four parts. The first set of these contribute to what is referred to as the Kompaneets equation (Kompaneets 1957) describing spectral distortions to the CMB (Bernstein & Dodelson 1990):

$$\begin{aligned} c_K^{(2)}(\mathbf{p},\mathbf{p}') = \left(1+\cos^2\theta\right)\frac{(\mathbf{p}-\mathbf{p}')^2}{2m_e}\Bigg[&\left(f^{(0)}(p')-f^{(0)}(p)\right)T_e\frac{\partial^2\delta(p-p')}{\partial p'^2} \\ &- \left(f^{(0)}(p')+f^{(0)}(p)+2f^{(0)}(p')f^{(0)}(p)\right)\frac{\partial\delta(p-p')}{\partial p'}\Bigg] \\ +\frac{2(p-p')\cos\theta(1-\cos^2\theta)}{m_e}\Bigg[&\delta(p-p')f^{(0)}(p')(1+f^{(0)}(p)) \\ &- \left(f^{(0)}(p')-f^{(0)}(p)\right)\frac{\partial\delta(p-p')}{\partial p'}\Bigg]. \end{aligned} \tag{2.13}$$

There is also the simple damping term

$$c_{\Delta}^{(2)}(\mathbf{p},\mathbf{p}') = \left(1+\cos^2\theta\right)\delta(p-p')(f^{(2)}(\mathbf{p}')-f^{(2)}(\mathbf{p})); \tag{2.14}$$

a set of terms coupling the photon perturbation to the velocity

$$c_{\Delta v}^{(2)}(\mathbf{p},\mathbf{p}') = \left(1+\cos^2\theta\right)\left(f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p})\right)\Bigg[\left(1+\cos^2\theta\right)(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}\frac{\partial\delta(p-p')}{\partial p'}$$

$$- \ 2\cos\theta(1-\cos\theta)\delta(p-p')(\hat{\mathbf{p}}+\hat{\mathbf{p}}')\cdot\mathbf{v}\Bigg]; \tag{2.15}$$

and finally a set of source terms quadratic in the velocity:

$$c_{vv}^{(2)}(\mathbf{p},\mathbf{p}') = \left(f^{(0)}(p')-f^{(0)}(p)\right)\ (\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}\Bigg[\left(1+\cos^2\theta\right)\frac{(\mathbf{p}-\mathbf{p}')\cdot\mathbf{v}}{2}\frac{\partial^2\delta(p-p')}{\partial p'^2}$$

$$- \ 2\cos\theta(1-\cos\theta)(\hat{\mathbf{p}}+\hat{\mathbf{p}}')\cdot\mathbf{v}\frac{\partial\delta(p-p')}{\partial p'}.\Bigg] \tag{2.16}$$

We'll see in the next section that the first order terms, those in Eq. (2.12), reduce to the standard first order equation once the p' and Ω' integrals are done. There are a lot of second order terms. As we have mentioned, one subset of these terms has already been studied: those leading to the Kompaneets equation (Kompaneets 1957). We know that Vishniac analyzed a second order term. Which one is it here? It turns out that he looked at none of the second order terms in $c^{(2)}$ [or at least he didn't write a paper about any of them]. Rather, he got yet another second order term by expanding the electron density as $n_e = \bar{n_e}(1+\delta_e)$ and then multiplying δ_e by the first order terms in Eq. (2.12).

3. First Order Equation

In this section, we derive the well-known first order equation coupling photons and electrons. To do this, we need focus only on the terms in Eq. (2.12). For the angular integrals, we choose the polar axis to lie along the direction of the electron velocity, so that azimuthal symmetry is maintained. Then μ' is the polar angle defined by $\mu' \equiv \hat{\mathbf{v}}\cdot\hat{\mathbf{p}}'$; we also define $\mu \equiv \hat{\mathbf{v}}\cdot\hat{\mathbf{p}}$. Thus we have

$$C^{(1)}(\mathbf{p}) = \frac{3n_e\sigma_T}{4p}\int dp'p'\frac{d\Omega'}{4\pi}\,c^{(1)}(\mathbf{p},\mathbf{p}')$$

$$= \frac{3n_e\sigma_T}{4p}\int_0^\infty dp'p'\Bigg[\delta(p-p')\int_{-1}^{1}\frac{d\mu'}{2}\,(f^{(1)}(\mathbf{p}')-f^{(1)}(\mathbf{p}))\int_0^{2\pi}\frac{d\phi'}{2\pi}(1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2)$$

$$+ v(f^{(0)}(p')-f^{(0)}(p))\frac{\partial\delta(p-p')}{\partial p'}\int_{-1}^{1}\frac{d\mu'}{2}\,(p\mu-p'\mu')\int_0^{2\pi}\frac{d\phi'}{2\pi}(1+(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')^2)\Bigg], \tag{3.1}$$

where $v \equiv |\mathbf{v}|$. The dot product $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$ is a function of both μ' and ϕ'. But $f^{(1)}(\mathbf{p}')$ depends only on p' and μ', not on ϕ' because of the azimuthal symmetry. [This is perhaps too strong a statement. For many metrics considered by cosmologists, there *is* an azimuthal symmetry so $f^{(1)}$ depends only on the polar angle. There are cosmologies though wherein this symmetry is not maintained, so $f^{(1)}$ could well depend on the azimuthal angle. This happens for example when tensor perturbations (Crittenden *et al.* 1993) or cosmic strings are present. We will restrict our analysis to cases where the symmetry exists.] To do the ϕ' integral we first rewrite the integrand in terms of Legendre polynomials

$$+(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')^2 = \frac{4}{3}\left(1 + \frac{1}{2}P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\right) \tag{3.12}$$

By the addition theorem of spherical harmonics,

$$P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}'}) = \sum_{m=-2}^{m=2} \frac{(2-m)!}{(2+m)!} P_2^m(\hat{\mathbf{p}} \cdot \hat{\mathbf{v}}) P_2^m(\hat{\mathbf{p}'} \cdot \hat{\mathbf{v}}) e^{im(\phi' - \phi)} \tag{3.13}$$

Thus,

$$\int \frac{d\phi'}{2\pi} P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}'}) = P_2(\hat{\mathbf{p}} \cdot \hat{\mathbf{v}}) P_2(\hat{\mathbf{p}'} \cdot \hat{\mathbf{v}}) \equiv P_2(\mu) P_2(\mu') \tag{3.14}$$

To do the μ' integrals, it is useful to define the moments of the distribution function:

$$f_l(p) = \int_{-1}^{1} \frac{d\mu}{2} P_l(\mu) f(p, \mu). \tag{3.15}$$

Now we can use the orthonormality of the Legendre polynomials to write

$$\int_{-1}^{1} \frac{d\mu'}{2} \left(1 + P_2(\mu) P_2(\mu')\right) \left[f^{(1)}(\mathbf{p}') - f^{(1)}(\mathbf{p})\right] = f_0^{(1)}(p') + \frac{1}{2} f_2^{(1)}(p') P_2(\mu) - f^{(1)}(p, \mu) \tag{3.16}$$

The notation at this stage is a bit confusing, so let's restate it: the superscript refers to the order of perturbation theory. Here we are considering the first order correction. The subscript refers to the moment of the distribution function; note that the last term in Eq. (3.16) has not been integrated over, *i.e.* still depends on μ, so it has no subscript. Eq. (3.1) now becomes

$$\begin{aligned} C^{(1)}(\mathbf{p}) =& \frac{n_e \sigma_T}{p} \int_0^\infty dp' p' \Bigg[\delta(p - p') \left(f_0^{(1)}(p') + \frac{1}{2} f_2^{(1)}(p') P_2(\mu) - f^{(1)}(p, \mu) \right) \\ & + v p \mu \left(f^{(0)}(p') - f^{(0)}(p) \right) \frac{\partial \delta(p - p')}{\partial p'} \Bigg]. \end{aligned} \tag{3.17}$$

The remaining p' integral can be done, in the first case trivially and in the second by integrating by parts. Thus our final expression for the collision term is

$$C^{(1)}(p,\mu) = n_e\sigma_T\left[f_0^{(1)} + \frac{1}{2}f_2^{(1)}P_2(\mu) - f^{(1)} - p\frac{\partial f^{(0)}}{\partial p}\mu v\right]. \tag{3.18}$$

It appears as if this collision term is momentum dependent. To get rid of this illusion, it is useful to define

$$\Delta(\mu) \equiv -\left[\frac{p}{4}\,\frac{\partial f^{(0)}}{\partial p}\right]^{-1} f^{(1)}(p,\mu). \tag{3.19}$$

Then combining the left hand side of the Boltzmann equation from Eq. (2.2) and the right hand from Eq. (3.18), the full first order equation is

$$\left\{\frac{\partial}{\partial t} + \frac{\mathbf{p}}{p}\cdot\frac{\partial}{\partial \mathbf{x}}\right\}\Delta = n_e\sigma_T\Big(\Delta_0 + \frac{1}{2}\Delta_2 P_2(\mu) + 4\mu v - \Delta\Big) \tag{3.20}$$

[The expansion term in Eq. (2.2) simply forces the zero order distribution function to depend on the *comoving* momentum pa where a is the scale factor. One can then show that the expansion term drops out of the first order equation for Δ.] The subscripts here again refer to the moments of $\Delta(\mu)$ defined just as in Eq. (3.15). Since to first order Δ is independent of photon energy, p, Δ as defined by Eq. (3.19) is equal to the brightness defined as

$$\Delta = \frac{\int dp p^3 f^{(1)}}{\int dp p^3 f^{(0)}}. \tag{3.21}$$

4. First Order Solutions

Now that we have the equations describing the interactions of photons with electrons, we can solve them to determine the predicted anisotropies in the cosmic microwave background. Figure 2 shows the results of numerically integrating the full set of linear equations starting with these initial conditions, assuming standard recombination at $z \simeq 1100$. We'd like to do two things in this section: First, we would like to understand the peak at $l \simeq 200$. If we understand why it occurs at $l \sim 200$ when the Universe follows a standard ionization history, then we will be able to understand how this peak shifts when we consider different ionization histories. Second, we would like to understand the damping that occurs at $l \sim 1000$. We will see that this damping is due to the "finite thickness of the last scattering surface." For a re-ionized Universe, this thickness is much larger, and therefore, damping is apparent even on very large scales [much smaller l]. It turns out that understanding the physical reason for this damping will give us a clue as to which second order terms are likely to be significant. [Due to the crudeness of our approximations, we will not be able to account for the oscillations that occur as C_l is damped; these are due to acoustic waves at the time of recombination.]

To understand these features, we will solve a simplified version of Eq. (3.20). First note that Δ depends on position; to account for this, we can Fourier transform it:

$$\Delta(\mathbf{x}, \mathbf{p}, \tau) = V \int d^3k \; \tilde{\Delta}(k, \mu, \tau) e^{i\mathbf{k}_{\text{physical}} \cdot \mathbf{x}}, \tag{4.1}$$

where $k_{\text{physical}} = k/a$; k being the comoving wavenumber; a the cosmic scale factor; V the volume, a factor which drops out of all physical results. Note that because of the azimuthal symmetry, $\tilde{\Delta}$ depends only on the magnitude of k and the dot product $\hat{\mathbf{k}} \cdot \hat{\mathbf{p}} = \tilde{\mathbf{v}} \cdot \hat{\mathbf{p}} = \mu$. Note that the first equality holds here since the velocity is irrotational, meaning that the Fourier transform of the velocity is parallel to $\mathbf{k}$. We also define the conformal time

$$\tau \equiv \int_0^t \frac{dt'}{a(t')}. \tag{4.2}$$

The conformal time today is

$$\tau_0 = \frac{2}{H_0} = 1.2 \times 10^{18} \text{ sec} \tag{4.3}$$

since we're assuming a flat, matter, dominated Universe with $h = 0.5$. The first order equation is now

$$\dot{\tilde{\Delta}} + ik\mu\tilde{\Delta} = n_e\sigma_T a \left(4\mu\tilde{v} - \tilde{\Delta}\right) \tag{4.4}$$

where the dot denotes derivative with respect to τ. The Δ_0 and Δ_2 terms on the right hand side of Eq. (3.20) have been neglected. That's because we are only interested in the kick that photons get from the moving electrons [the $4\mu\tilde{v}$ term], the so-called *Doppler effect.* The Δ_0, Δ_2 terms give rise to what's sometimes called the *intrinsic* anisotropy. Also we still have not written down the terms that represent the perturbation to the metric. These lead to the Sachs-Wolfe effect. So all we should get out of the simplified Eq. (4.4) is the Doppler effect.

Eq. (4.4) is a first order differential equation, whose solution is

$$\begin{aligned}\tilde{\Delta}(k, \mu, \tau) =& \tilde{\Delta}(k, \mu, \tau_{\text{initial}}) \exp\left\{ik\mu(\tau_{\text{initial}} - \tau) - \int_{\tau_{\text{initial}}}^{\tau} d\tau' n_e(\tau')\sigma_T a(\tau')\right\} \\ &+ 4\mu \int_{\tau_{\text{initial}}}^{\tau} d\tau' n_e(\tau')\sigma_T a(\tau')\tilde{v}(k, \tau') e^{ik\mu(\tau' - \tau)} \\ &\times \exp\left\{-\int_{\tau'}^{\tau} d\tau'' n_e(\tau'')\sigma_T a(\tau'')\right\}.\end{aligned} \tag{4.5}$$

The first term on the right represents the anisotropies that were initially present and have persisted until some late time τ. These are damped out by scattering with electrons, the $\int n_e\sigma_T a d\tau'$ term in the exponential. If we choose an early enough τ_{initial}, then this integral is always large, and so the initial anisotropies are not important today. We can write Eq. (4.5) in a more compact form by setting $\tau_{\rm initial}$ to zero and by defining the *visibility function*

$$g(\tau,\tau') \equiv n_e(\tau')\sigma_T a(\tau') \exp\left\{-\int_{\tau'}^{\tau} d\tau'' n_e(\tau'')\sigma_T a(\tau'')\right\}. \tag{4.6}$$

Then we have simply

$$\tilde{\Delta}(k,\mu,\tau) = 4\mu \int_0^{\tau} d\tau' \tilde{v}(k,\tau') g(\tau,\tau') e^{ik\mu(\tau'-\tau)}. \tag{4.7}$$

The visibility function defined in Eq. (4.6) has an interesting physical interpretation. To see this, first note that $\int_0^\infty d\tau' g(\infty,\tau') = 1$, so g is normalized like a probability density. In fact, that's what it is: the probability per unit conformal time that a photon at time τ was last scattered at time τ'. The visibility function depends on the free electron density n_e; thus it is particularly sensitive to the ionization history. Figure 3 shows the visibility function for two ionization histories: (i) standard recombination at $z \sim 1100$ and (ii) the case where the electrons remain ionized throughout the whole history of the Universe. In the first case recombination happens rapidly, so the "surface of last scattering" is centered at $z = 1100$ but with a very small width. In the second case, the surface of last scattering is centered at $z = 100$ but is quite wide. To simplify Eq. (4.7) further, we can approximate g as a Gaussian:

$$g(\tau_0,\tau') \simeq \frac{1}{\sqrt{\pi}\delta\tau_R} \exp\left\{-\frac{(\tau'-\tau_R)^2}{\delta\tau_R^2}\right\}. \tag{4.8}$$

For standard recombination, $\tau_R \simeq .02\tau_0$ while the width of the last scattering surface, $\delta\tau_R \simeq .1\tau_R$. For no recombination, $\tau_R \sim .08\tau_0$ while $\delta\tau_R \sim .06\tau_0$.

Our goal is to find out how $\tilde{\Delta}$ depends on the ionization parameters, τ_R and $\delta\tau_R$. With the approximation in Eq. (4.8) we can go further and solve explicitly for $\tilde{\Delta}$ in Eq. (4.7). We now have

$$\tilde{\Delta}(k,\mu,\tau_0) = 4\mu \frac{1}{\sqrt{\pi}\delta\tau_R} \int_0^{\tau_0} d\tau' \tilde{v}(k,\tau') e^{ik\mu(\tau'-\tau_0)} \exp\left\{-\frac{(\tau'-\tau_R)^2}{\delta\tau_R^2}\right\}. \tag{4.9}$$

In a matter dominated Universe, $\tilde{v}$ grows as τ, so $\tilde{v}(\tau) = \tilde{v}(\tau_0)\tau/\tau_0$. We can also set the upper limit of the integral to infinity since the exponential is negligible for $\tau > \tau_0$. Then,

$$\tilde{\Delta}(k,\mu,\tau_0) = \frac{4\mu\tilde{v}(k,\tau_0)}{\sqrt{\pi}\tau_0\delta\tau_R} e^{-ik\mu\tau_0} \frac{1}{i}\frac{\partial}{\partial(k\mu)} \int_0^\infty d\tau' e^{ik\mu\tau'} \exp\left\{-\frac{(\tau'-\tau_R)^2}{\delta\tau_R^2}\right\}. \tag{4.10}$$

where we have written τ' as $(1/i)\partial/\partial(k\mu)$. The integral in Eq. (4.10) contains a lot of the physics we are after.

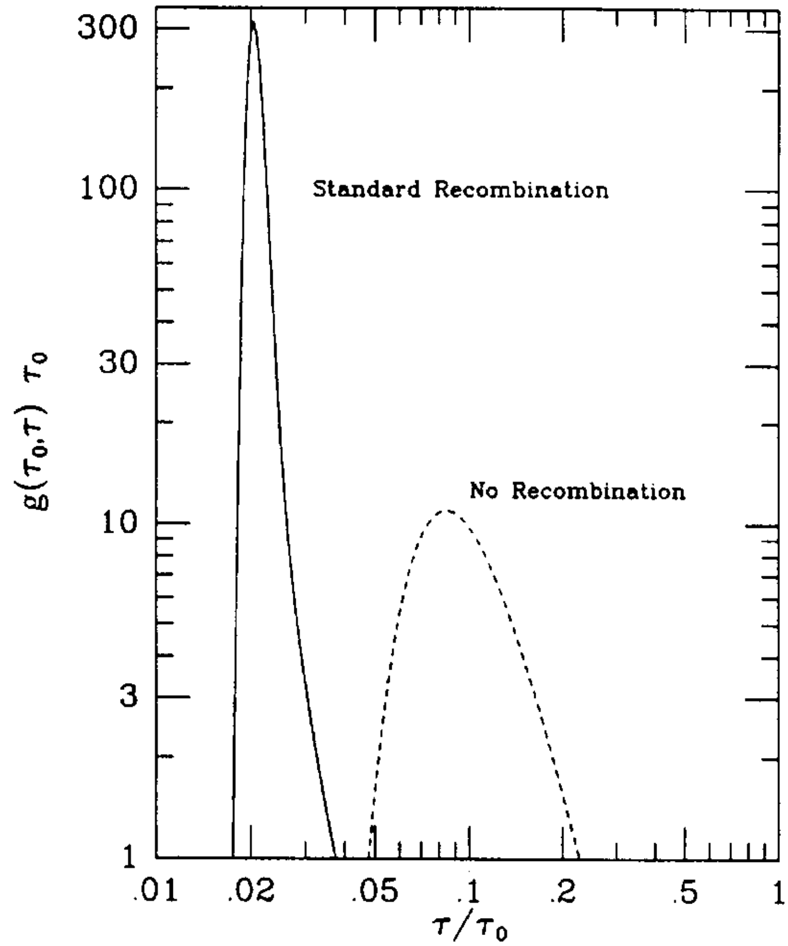

FIG. 3. The visibility functions for standard recombination [solid line] and no recombination [dashed line].

Figure 4 shows the integrand for two limiting cases: (i) $k\mu\delta\tau_R >> 1$ and (ii) $k\mu\delta\tau_R << 1$. In the first case [large k], the scale of the perturbation is much smaller than $\delta\tau_R$, the width of the surface of last scattering. A photon travelling through the last scattering surface travels through many regions where v is positive but an almost equal number of regions where v is negative. Thus the total contribution to $\tilde{\Delta}$ is small on scales smaller than the thickness of the last scattering surface. The second case comes about either because k is small [i.e. the scale of the perturbation is large] or $\mu \sim 0$ [$\hat{\mathbf{p}} \cdot \hat{\mathbf{v}} \propto \vec{p} \cdot \nabla\delta \sim 0$, the photon is travelling perpendicular to the direction in which the perturbation is changing]. Perturbations on large scales *do* make a large contribution to $\tilde{\Delta}$ since there is no cancellation through the last scattering surface. Similarly, there is no cancellation [in the integral] if the photon is travelling perpendicular to the gradient of the perturbation. [Note though that Eq. (4.10) has a factor of μ in front which does lead to a cancellation for such photons.] In any event, the integral under discussion can be written down in terms of Error functions, but for our purposes we will make another approximation which will simplify things further. The contribution of the integrand at the lower limit $\tau' = 0$ is suppressed by a factor of $e^{-(\tau_R/\delta\tau_R)^2}$; for most realistic ionization histories this will be a

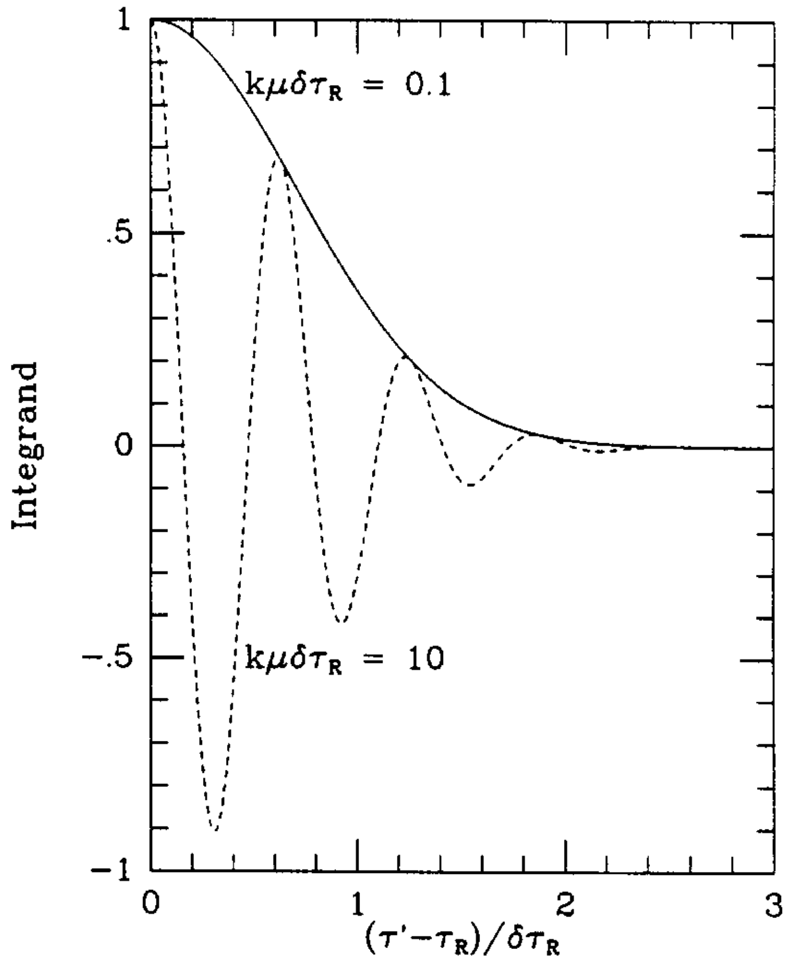

FIG. 4. Real part of the integrand of Eq. (4.10).

pretty small number, so we can extend the lower limit all the way to $-\infty$ with little loss of accuracy. The remaining integral is then the Fourier transform of a Gaussian, which is itself a Gaussian, or

$$\tilde{\Delta}^{\text{Doppler}}(k,\mu,\tau_0) = 4\mu\tilde{v}(k,\tau_0)\,\frac{\tau_R}{\tau_0}\left[1 + i\frac{k\mu\delta\tau_R^2}{2\tau_R}\right]e^{ik\mu(\tau_R-\tau_0)}e^{-(k\mu\delta\tau_R/2)^2}, \tag{4.11}$$

where the superscript indicates that this is the perturbation to the photons caused by the Doppler effect. Eq. (4.11) clearly illustrates the damping on small scales as we have discussed; the damping scale is of order $k \sim 1/\delta\tau_R$. Kaiser (1984) pointed out that since the velocity is parallel to **k**, there is an extra factor of μ in front of Eq. (4.11). Thus, for example, $\delta\rho_\gamma/\rho_\gamma = (1/2)\int_{-1}^{1} d\mu\ \tilde{\Delta}^{\text{Doppler}}(k,\mu,\tau_0) \propto 1/(k\delta\tau_R)^2$. This suppression of $\delta\rho_\gamma/\rho_\gamma$ would be averted if the source term had a component perpendicular to **k**. The other feature of Eq. (4.11) which is of interest is the factor of τ_R/τ_0 in front. This tells us that the later the photons are in contact with the electrons, the greater is the Doppler kick they get. But this is what we expect: velocities grow with τ, so photons scattering off electrons at later times are seeing larger velocities. This simple fact will have surprising consequences, namely, on large scales [before damping sets in] a reionized Universe produces a *larger* signal in the CMB than one which never reionizes!

What is the contribution of the Doppler effect to the present day anisotropy as measured by the C_l's? Referring back to Figure 2, we see that for $l < 30$ or so, the dominant contribution to C_l comes from the Sachs-Wolfe effect, for which $l(l+1)C_l$ is constant. At larger l's, the Doppler effect becomes relevant. The reason for deriving an analytic expression for $\tilde{\Delta}^{\rm Doppler}$ in terms of $\delta\tau_R$ and τ_R is to see when the Doppler effect becomes important as we vary the ionization history of the Universe. To calculate the C_l's, we expand $\tilde{\Delta}^{\rm Doppler}(k,\mu,\tau_0)$ in a series of Legendre polynomials, defining

$$\tilde{\Delta}_l^{\rm Doppler}(k,\tau_0) = \frac{1}{2}\int_{-1}^{1} d\mu\; P_l(\mu)\tilde{\Delta}^{\rm Doppler}(k,\mu,\tau_0). \tag{4.12}$$

Then, C_l is given by:

$$\begin{aligned} C_l^{\rm Doppler} &= \frac{V}{8\pi}\int_0^\infty dk\; k^2\;\langle\left|\tilde{\Delta}_l^{\rm Doppler}(k,\tau_0)\right|^2\rangle \\ &= \frac{V}{8\pi}\int_0^\infty dk\; k^2\;\langle\left|\frac{1}{2}\int_{-1}^{1} d\mu P_l(\mu) 4\mu\tilde{v}(k,\tau_0)\left[\frac{\tau_R}{\tau_0} + i\frac{k\mu\delta\tau_R^2}{2\tau_0}\right]\right. \\ &\qquad \left.\times\; e^{ik\mu(\tau_R-\tau_0)}e^{-(k\mu\delta\tau_R/2)^2}\right|^2\rangle. \end{aligned} \tag{4.13}$$

Here we are using the normalization of Efstathiou, Bond, & White (1992). [A good way to check normalization is to insure that $C_2 = (8/\pi\tau_0^4)\int_0^\infty dk j_2^2(k)P(k)/k^2$, where $P(k)$ the power spectrum is normalized to COBE, so on large scales is proportional to $C_2 k$.] As a first approximation to $C_l^{\rm Doppler}$, we'll assume that only $k < 2/\delta\tau_R$ contribute to the $k-$ integral due to damping, and that for these values of k we can set the exponential $e^{-(k\mu\delta\tau_R/2)^2}$ to one. Then the integral over μ is simply the derivative of a spherical Bessel function:

$$\frac{1}{2}\int_{-1}^{1} d\mu\mu P_l(\mu)e^{ik\mu(\tau_R-\tau_0)} = \frac{1}{i}\frac{\partial}{\partial(k(\tau_R-\tau_0))}\frac{j_l(k(\tau_R-\tau_0))}{(-i)^l}, \tag{4.14}$$

and similarly the term with μ^2 becomes the second derivative of a spherical Bessel function. Also the continuity equation tells us that $\tilde{\mathbf{v}}(k,\tau_0) = -i\mathbf{k}\dot{\tilde{\delta}}_e(k,\tau_0)/k^2 = -2i\mathbf{k}\tilde{\delta}_e(k,\tau_0)/k^2\tau_0$, where $\tilde{\delta}_e$ is the fractional change in density of the electrons, assumed equal to that of the rest of the matter. Thus, the ensemble average is:

$$< |\tilde{v}(k,\tau_0)|^2 >= \frac{4}{(k\tau_0)^2}\;\; < |\tilde{\delta}_e(k,\tau_0)|^2 >= \frac{4P(k)}{V(k\tau_0)^2}. \tag{4.15}$$

Here $P(k)$ is the COBE normalized power spectrum;for CDM (Efstathiou *et al.* 1992), $P_{CDM}(k) = \frac{3\pi}{2}C_2\tau_0^4 kT^2(k)$ where $C_2 = 4\pi(Q_{RMS}/T_0)^2/5 = 9.7\times 10^{-11}$ and T is the transfer function.

We now have our final result for $C_l^{\rm Doppler}$ in terms of a simple one dimensional integral:

$$C_l^{\rm Doppler} \simeq \frac{8}{\pi}\left(\frac{\tau_R}{\tau_0}\right)^2\int_0^{2\tau_0/\delta\tau_R} dz\;\frac{P(k=z/\tau_0)}{\tau_0^3}\;\left[\frac{dj_l(z)}{dz} - \frac{\delta\tau_R^2}{2\tau_0\tau_R}z\frac{d^2j_l(z)}{dz^2}\right]^2. \tag{4.17}$$

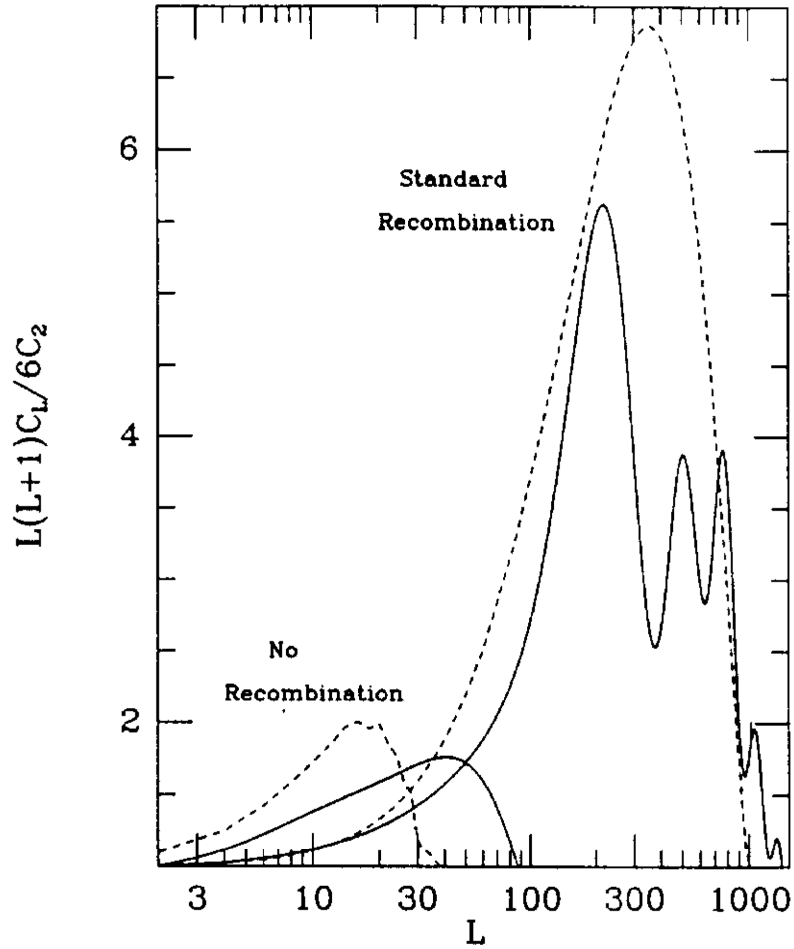

FIG. 5. $l(l+1)C_l$ for standard recombination and no recombination. The results of a full numerical solution of the coupled Boltzmann equations [solid lines] are compared with the approximate solution given in Eq. (4.17) [dashed lines].

Figure 5 shows this approximate result for $C_l^{\rm Doppler}$ added to the Sachs-Wolfe result of $C_l^{\rm Sachs-Wolfe} = 6C_2/l(l+1)$ for standard recombination and no recombination. Also shown are the results of numerically integrating the full set of Boltzmann equations [including perturbation to the metric, CDM and three species of massless neutrinos] for these two ionization histories. Apparently the approximate result of Eq. (4.16) gives a very good qualitative picture of the results we are after: (i) the Doppler peak *is* larger on large scales [small l] and (ii) the damping *does* take place at larger scales than in the case of standard recombination. Clearly, though, for an accurate quantitative analysis the equations must be solved numerically. One final quantitative point: Having emphasized the fact that C_l is larger in a no-recombination Universe at low l, we should point out that it's not *that* much larger. The difference never exceeds 30% for any l. Thus we'll see in Section 7 that the expected ΔT [which goes as the square root of C_l] for no-recombination is roughly the same on these scales as for standard recombination. All our work in this section has been on the linear terms in the Boltzmann equation. We now turn to second order terms.

5. Second Order Equation

In this section we write down the second order equation which follows from our general Boltzmann treatment in Section 2. It turns out that there are numerous terms. We can immediately throw out one class of terms in Eq. (2.11) though. The terms coupling $f^{(1)}$ and v, i.e. those in Eq. (2.15), are nominally second order. In practice, though, we have seen that $f^{(1)}$ [or its equivalent $\Delta^{(1)}$] is very small on small scales due to damping. So we we can neglect $c^{(2)}_{\Delta v}$ in Eq. (2.11). The remaining terms can be manipulated as were the first order terms in Section 3. The result for the second order collision term is

$$C^{(2)}(\mathbf{p}) = n_e\sigma_T\Bigg\{ -p\frac{\partial f^{(0)}}{\partial p}\delta_e\mu v \;+\; f_0^{(2)} + \frac{1}{2}f_2^{(2)}P_2(\mu) - f^{(2)}$$
$$+ v^2 p\frac{\partial f^{(0)}}{\partial p}(\mu^2+1) + v^2p^2\frac{\partial^2 f^{(0)}}{\partial p^2}(\frac{11}{20}\mu^2 + \frac{3}{20})$$
$$+ \frac{1}{m_e^2}\frac{\partial}{\partial p}\left[p^4\left(T_e\frac{\partial f^{(0)}}{\partial p} + f^{(0)}(1+f^{(0)})\right)\right]\Bigg\} \tag{5.1}$$

The first term on the right hand side arises from writing the electron density as $n_e = \bar{n}_e(1+\delta_e)$ and then multiplying the first order collision terms (Eq. (3.18)) by δ_e [note that we have dropped all terms with $f^{(1)}$'s in them]. This is the so-called "Vishniac term". The remaining terms arise from the second order expansion in the collision integral, Eq. (2.11). At second order, electron-photon scattering is not purely elastic. Energy is transferred between the electrons and photons, inducing spectral distortions. In the limit of no anisotropy only the terms on the last line, the Kompaneets terms which give rise to spectral Sunyaev-Zel'dovich distortions, survive. Since the Kompaneets terms do not induce anisotropies, we shall neglect them. Finally, since we are interested in scales much smaller than the horizon, non-Newtonian gravitational effects can be ignored.

It would be nice to define a $\Delta^{(2)}$ just as we did $\Delta^{(1)}$ in Eq. (3.19) so that the second order equation is momentum independent. This is impossible though, because the second order equation is *momentum dependent*! Specifically the v^2 terms in Eq. (5.1) are momentum dependent. Thus in general $\Delta^{(2)}$ will depend on momentum. There are two ways to deal with this: Either we can go ahead and try to solve the full equation for $\Delta^{(2)}(k,p,\mu)$, or we can integrate out the momentum dependence. Let us first take the easy way out, and later we will see if it is necessary to include the full momentum dependence. To do the integration we can define

$$\Delta^{(2)} \equiv \frac{\int dpp^3 f^{(2)}}{\int dpp^3 f^{(0)}}, \tag{5.2}$$

as in Eq. (3.21). Then integrating Eq. (5.1) leads to the second order equation:

$$\frac{\partial \Delta^{(2)}}{\partial \tau} + a\hat{p}^i\frac{\partial \Delta^{(2)}}{\partial x^i} = a\bar{n}_e\sigma_T\left\{4\delta_e\hat{\mathbf{p}}\cdot\mathbf{v} + v^2\left(7 + 15(\hat{\mathbf{p}}\cdot\hat{\mathbf{v}})^2\right) - \Delta^{(2)}\right\}, \tag{5.3}$$

where we have dropped the $\Delta_0^{(2)}$ and $\Delta_2^{(2)}$ terms on the right hand side, as these should be irrelevant on small scales. On physical grounds, we can argue that the only important second order term is the "Vishniac" term. The Vishniac term is proportional to $\delta_e v$ whereas the other terms are proportional to v^2. By the continuity equation, though, $v \sim \delta_e/k\tau$. Since $k\tau$ is quite large (> 100) for the scales of interest, the Vishniac term should dominate.

Are the other terms in Eq. (5.3) completely irrelevant? Not necessarily. Recall that we integrated out the frequency dependence of the velocity squared terms in Eq. (5.1). Even if the final value of $< (\Delta T/T)^2 >$ induced by these terms was a factor of 100 smaller than the leading terms, we might be able to pick them out because of the frequency dependence. [A recent example of the process of separating out a signal from a much more powerful source with a different "spectral index," i.e. frequency dependence, can be found in Meinhold *et al.* (1993).] So we choose to keep these additional terms a little longer.

6. Second Order Contribution to $\frac{\Delta T}{T}$

In this section we will follow the treatment of Efstathiou (1988) in solving the second order equation. We can rewrite Eq. (5.3) as

$$\dot{\tilde{\Delta}}^{(2)} + ik\mu\tilde{\Delta}^{(2)} = \bar{n}_e\sigma_T a\left[S_{\delta v}(\mathbf{k},\tau) + S_{vv}(\mathbf{k},\tau) - \tilde{\Delta}^{(2)}\right] \tag{6.1}$$

where the Vishniac source term is

$$\begin{aligned} S_{\delta v}(\mathbf{k},\tau) &\equiv 4\hat{\mathbf{p}}\cdot\sum_{\mathbf{k}'}\tilde{\mathbf{v}}(\mathbf{k}',\tau)\tilde{\delta}_e(\mathbf{k}-\mathbf{k}',\tau) \\ &= \left(\frac{\tau}{\tau_0}\right)^3\frac{8i}{\tau_0}\sum_{\mathbf{k}'}\frac{\hat{\mathbf{p}}\cdot\mathbf{k}'}{k'^2}\,\tilde{\delta}_e(\mathbf{k}',\tau_0)\tilde{\delta}_e(\mathbf{k}-\mathbf{k}',\tau_0) \end{aligned} \tag{6.2}$$

and the source term quadratic in velocities is

$$\begin{aligned} S_{vv}(\mathbf{k},\tau) &\equiv \sum_{\mathbf{k}'}\left[7\tilde{\mathbf{v}}(\mathbf{k}',\tau)\cdot\tilde{\mathbf{v}}(\mathbf{k}-\mathbf{k}',\tau)\ +\ 15\hat{\mathbf{p}}\cdot\tilde{\mathbf{v}}(\mathbf{k}',\tau)\hat{\mathbf{p}}\cdot\tilde{\mathbf{v}}(\mathbf{k}-\mathbf{k}',\tau)\right] \\ &= \frac{-4(\tau/\tau_0)^2}{\tau_0^2}\sum_{\mathbf{k}'}\frac{7\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')\ +\ 15\hat{\mathbf{p}}\cdot\mathbf{k}'\hat{\mathbf{p}}\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2k'^2}\,\tilde{\delta}_e(\mathbf{k}',\tau_0)\tilde{\delta}_e(\mathbf{k}-\mathbf{k}',\tau_0). \end{aligned} \tag{6.3}$$

Here we have Fourier transformed from position space to momentum space. We have also used the facts that (i) the velocities are first order, and are related to the baryon density perturbations through the equation of continuity, and (ii) the time dependence of the perturbations is given by $\tilde{\delta}_e(\tau) = \tilde{\delta}_e(\tau_0)(\frac{\tau}{\tau_0})^2$ in a matter dominated Universe.

The second order equation is of the same form as the first order equation, whose solution we wrote down in Eq. (4.7). By analogy, the solution to the second order equation is

$$\tilde{\Delta}^{(2)}(k,\mu,\tau_0) = e^{-ik\mu\tau_0}\left[S_{\delta v}(\mathbf{k},\tau_0)I_1(k\mu\tau_0) + S_{vv}(\mathbf{k},\tau_0)I_2(k\mu\tau_0)\right] \tag{6.4}$$

where the time integrals are

$$I_1(k\mu\tau_0) = \int_0^{\tau_0} d\tau' \left(\frac{\tau'}{\tau_0}\right)^3 g(\tau_0, \tau') e^{ik\mu\tau'}$$

$$I_2(k\mu\tau_0) = \int_0^{\tau_0} d\tau' \left(\frac{\tau'}{\tau_0}\right)^2 g(\tau_0, \tau') e^{ik\mu\tau'} \tag{6.5}$$

with g our old friend, the visibility function. We encountered an integral of this form [only the powers of τ' differ] in section 4 when analyzing the first order equation. The main lesson we learned was that integrals of this type are strongly damped unless $k\mu$ is small. On small scales [large k] this means that the integrals are non-negligible only if $\mu \simeq 0$. Thus the main contribution to $\tilde{\Delta}^{(2)}$ can be found by evaluating the $S_{\delta v}(\tau_0)$ and $S_{vv}(\tau_0)$ at $\mu = 0$. Before doing this, let's go back to Eq. (4.7). There the source term was proportional to μ, so the contribution of the linear source term was greatly suppressed. Vishniac's profound observation and the origin of the dominance of second order terms is that the second order terms $S_{\delta v}$ and S_{vv} do *not* vanish at $\mu = 0$.

For the very small angle experiments where second order effects are important, there is a simple formula for the C_l's:

$$C_l = \frac{lV}{32\pi\tau_0^3} \int_{-1}^{1} d\mu \, \langle |\Delta(k = l/\tau_0, \mu)|^2 \rangle \tag{6.6}$$

where V is the volume which will drop out at the end of the calculation. To derive this, start with the small angle formula (Doroshkevich, Zel'dovich, & Sunyaev 1978):

$$C(\theta, \sigma) = V \int_0^\infty \frac{dk\, k^2}{64\pi^2} \int_{-1}^{+1} d\mu (1-\mu^2) \langle |\Delta(k, \mu, \tau_0)|^2 \rangle J_0\left(kR_c\theta(1-\mu^2)\right) \exp\left[-(kR_c\sigma)^2\right] \tag{6.7}$$

where θ is the angle between the two observing direction; σ is width of the Gaussian beam observing the temperature differences; and R_c is the comoving distance to the last scattering surface, $R_c \approx \tau_0$ in a flat universe with $z_c \gg 1$. Since $\langle |\Delta(k, \mu, \tau_0)|^2 \rangle$ is highly peaked around $\mu = 0$, we can set $\mu = 0$ everywhere else in the integrand. Meanwhile the product of J_0 and the exponential suppression is just the experimental window function referred to in Eq. (1.1). Changing variables to $l = k/\tau_0$ leads to

$$C(\theta, \sigma) = \frac{V}{64\pi^2\tau_0^3} \int_0^\infty dl\, l^2 \int_{-1}^{+1} d\mu \, \langle |\Delta(k = l/\tau_0, \mu)|^2 \rangle W_{l,expt}. \tag{6.8}$$

Comparing this to the continuous form of Eq. (1.1) leads to Eq. (6.6). Let us now calculate the ensemble average $\langle |\Delta(k = l/\tau_0, \mu)|^2 \rangle$:

$$\begin{aligned} \langle |\Delta(k, \mu, \tau_0)|^2 \rangle = &< |S_{\delta v}|^2 > |I_1|^2 + < S_{\delta v} S_{vv}^* > I_1 I_2^* + < S_{\delta v}^* S_{vv} > I_1^* I_2 \\ &+ < |S_{vv}|^2 > |I_2|^2. \end{aligned} \tag{6.9}$$

We argued above that $S_{\delta v}/S_{vv} < 10^{-2}$, so the last term here is suppressed by a factor of at least 10^{-4} relative to the first and we can safely drop it. The only terms with S_{vv} that might be interesting are the cross terms, so we will keep these. Upon squaring S, we encounter double sums, say over $\mathbf{k}'$ and $\mathbf{k}''$; thus we need the identity

$$< \tilde{\delta_e}^*(\mathbf{k}')\tilde{\delta_e}^*(\mathbf{k}-\mathbf{k}')\tilde{\delta_e}(\mathbf{k}'')\tilde{\delta_e}(\mathbf{k}-\mathbf{k}'') >= \frac{P(\mathbf{k}')P(\mathbf{k}-\mathbf{k}')}{V}\left[\delta_{\mathbf{k}',\mathbf{k}''} + \delta_{\mathbf{k}'+\mathbf{k}'',\mathbf{k}}\right] \tag{6.10}$$

where P is the power spectrum today and δ, the Kronecker delta. We can use this expression to expand Eq. (6.9) as

$$\langle|\Delta(k,\mu,\tau_0)|^2\rangle = \frac{64}{V^2\tau_0^2}\sum_{\mathbf{k}',\mathbf{k}''} P(\mathbf{k}')P(\mathbf{k}-\mathbf{k}')\left[\delta_{\mathbf{k}',\mathbf{k}''} + \delta_{\mathbf{k}'+\mathbf{k}'',\mathbf{k}}\right]\frac{\hat{\mathbf{p}}\cdot\mathbf{k}''}{k''^2}$$
$$\times\left\{\frac{\hat{\mathbf{p}}\cdot\mathbf{k}'}{k'^2}|I_1|^2 + \frac{7\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}') + 15\hat{\mathbf{p}}\cdot\mathbf{k}'\hat{\mathbf{p}}\cdot(\mathbf{k}-\mathbf{k}')}{k'^2|\mathbf{k}-\mathbf{k}'|^2\tau_0}\mathrm{Im}(I_1I_2^*)\right\} \tag{6.11}$$

To go further we use the Kronecker deltas to get rid of the $\mathbf{k}''$ sum and change the $\mathbf{k}'$ sum into an integral via $\sum_{\mathbf{k}'} \to V\int d^3k'/(2\pi)^3$. Furthermore, due to the damping effect we can set $\hat{\mathbf{p}}\cdot\mathbf{k} = \mu = 0$ everywhere except in the time integrals (I_1, I_2). Thus,

$$\langle|\Delta(k,\mu,\tau_0)|^2\rangle = \frac{64}{V\tau_0^2(2\pi)^3}\int\frac{d^3k'}{k'^2}\,P(\mathbf{k}')P(\mathbf{k}-\mathbf{k}')\;\hat{\mathbf{p}}\cdot\mathbf{k}'\,\left(\frac{1}{k'^2} - \frac{1}{|\mathbf{k}-\mathbf{k}'|^2}\right)$$
$$\times\left\{\hat{\mathbf{p}}\cdot\mathbf{k}'|I_1|^2 + \frac{7\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}') - 15(\hat{\mathbf{p}}\cdot\mathbf{k}')^2}{|\mathbf{k}-\mathbf{k}'|^2\tau_0}\mathrm{Im}(I_1I_2^*)\right\}. \tag{6.12}$$

With no loss of generality we orient our coordinate system such that:

$$\mathbf{k}\cdot\hat{\mathbf{p}} = 0;\quad \mathbf{k}\cdot\mathbf{k}' = kk'x;\quad \mathbf{k}'\cdot\hat{\mathbf{p}} = k'\sqrt{(1-x^2)}\sin\phi \tag{6.13}$$

Then

$$\langle|\Delta(k,\mu,\tau_0)|^2\rangle = \frac{64}{V\tau_0^2(2\pi)^3}\int_0^\infty dk'\int_{-1}^1 dx\int_0^{2\pi} d\phi\;\;P(k')P\Big((k^2+k'^2-2kk'x)^{1/2}\Big)$$
$$\times\, k'\sqrt{1-x^2}\sin\phi\,\left(\frac{1}{k'^2} - \frac{1}{k^2+k'^2-2kk'x}\right)$$
$$\times\left\{k'\sqrt{1-x^2}\sin\phi|I_1|^2 + \frac{7(kk'x-k'^2) - 15k'^2(1-x^2)\sin^2\phi}{(k^2+k'^2-2kk'x)\tau_0}\mathrm{Im}(I_1I_2^*)\right\}. \tag{6.14}$$

We now see that the cross terms – those with $I_1 I_2$ – vanish once the ϕ integral is done (Hu *et al.* 1993). Thus we conclude that *the only second order term of any significance is the Vishniac term.* Plugging this expression into Eq. (6.6) yields

$$C_l = \frac{lJP^2(k = l/\tau_0)\ K(k = l/\tau_0)}{2\pi^2\tau_0^6} \tag{6.15}$$

where

$$K(k) = \frac{1}{k}\int_0^\infty dk' \int_{-1}^{+1} dx \frac{P(k')P\Big((k^2 + k'^2 - 2kk'x)^{1/2}\Big)}{P^2(k)} \frac{(1 - x^2)(k^2 - 2xkk')}{(k^2 + k'^2 - 2xkk')} \tag{6.16}$$

and

$$J \equiv \frac{k\tau_0}{2\pi}\int_{-1}^{1} d\mu\ |I_1(k\mu\tau_0)|^2. \tag{6.17}$$

For CDM, K, the integral over the power spectrum, can be approximated by $K(l) \simeq -0.1 + 1.1(l/1000)$ for $1000 < l < 5000$, with a maximum error of 5%. [The integral K is half of the integral Efstathiou (1988) calls I_2. It's also much easier to compute numerically. To prove the identity, one uses the invariance of the integrand under $\mathbf{k}' \leftrightarrow \mathbf{k} - \mathbf{k}'$.]

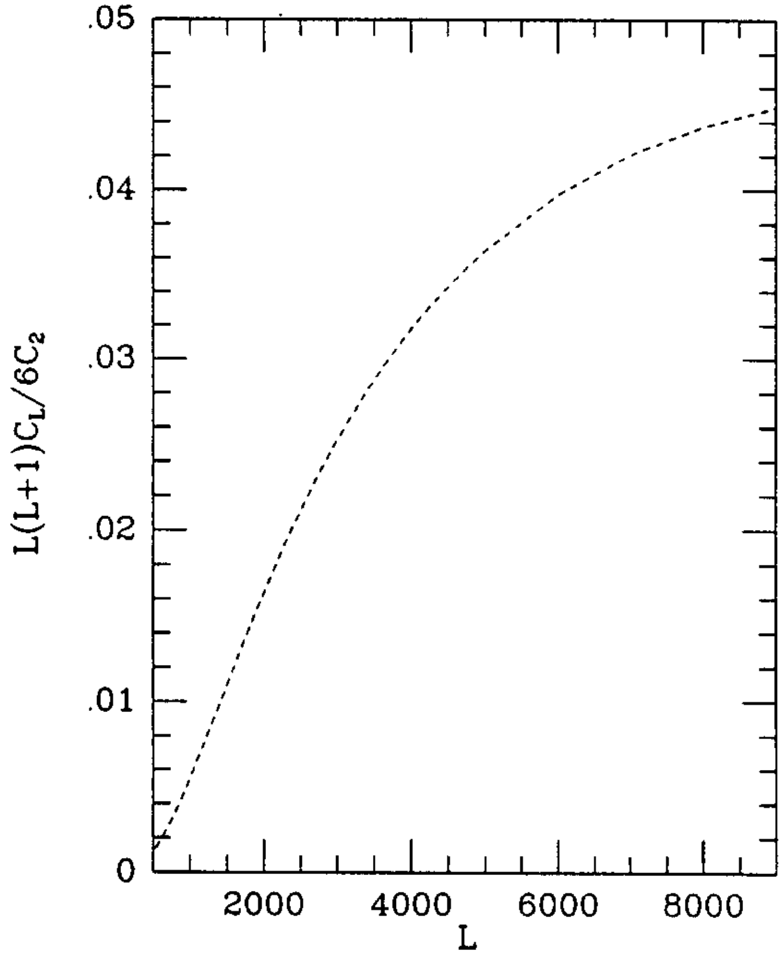

FIG. 6. C_l's induced by the second order Vishniac term for CDM with no recombination.

The integral J contains all the information about the ionization history, but it also appears to depend on wavenumber k. Efstathiou (1988) made the intriguing observation that for most reasonable ionization histories, J is in fact independent of k. To see this, we note that the integrand is sharply peaked around $\mu = 0$ so there should be very little error introduced if we extend the limits of integration all the way to $\pm\infty$. Then,

$$J \simeq \frac{1}{2\pi}\int_{-\infty}^{\infty} d(k\mu\tau_0)\int_0^{\tau_0} d\tau' \left(\frac{\tau'}{\tau_0}\right)^3 g(\tau_0,\tau')e^{ik\mu\tau'}\int_0^{\tau_0} d\tau'' \left(\frac{\tau''}{\tau_0}\right)^3 g(\tau_0,\tau'')e^{-ik\mu\tau''}$$
$$= \int_0^{\tau_0} d\tau'\left(\frac{\tau'}{\tau_0}\right)^6 g^2(\tau_0,\tau')\tau_0 \qquad (6.18)$$

where the last equality follows immediately since the $(k\mu\tau_0)$ integral yields a delta function in $\tau' - \tau''$. All the information about the ionization history is in the integral J. From Eq. (6.18), we see that the integrand is heavily towards late times by the τ'^6 factor. Thus ionization histories wherein the visibility function is peaked at late times – that is, scenarios in which the Universe is re-ionized – are most likely to produce appreciable secondary anisotropies. For standard recombination, $J = 1.7\times10^{-8}$ while $J = 7.3\times10^{-5}$ for no recombination.

Figure 6 shows the C_l's for CDM with no recombination. The shape of the curve is exactly the same for other ionization histories, only the amplitude, which is determined by J, drops. Nonetheless, since the integrand of J is heavily weighted towards late times, even relatively late re-ionization produces a comparable signal. The other feature of note in Fig. 6 is the amplitude, which is small. We'll see in the next section how this translates into an expected ΔT in a given experiment, but clearly it will be difficult to detect these secondary anisotropies in CDM. Other cosmologies, in particular baryon isocurvature models, are more likely to produce a large secondary signal due to the Vishniac effect (Bond & Efstathiou 1987; Efstathiou 1988; Hu *et al.* 1993). Although we won't calculate the signal in these models here, we stress that our conclusion that Vishniac's term is the only important second order one applies in general.

7. Results and Discussion

Now that we have all the physics under our belts, it is time to probe different ionization histories. In this section we present the C_l's for a variety of ionization histories and convolve them with the filter functions shown in Figure 1 to obtain the predicted signal in these experiments.

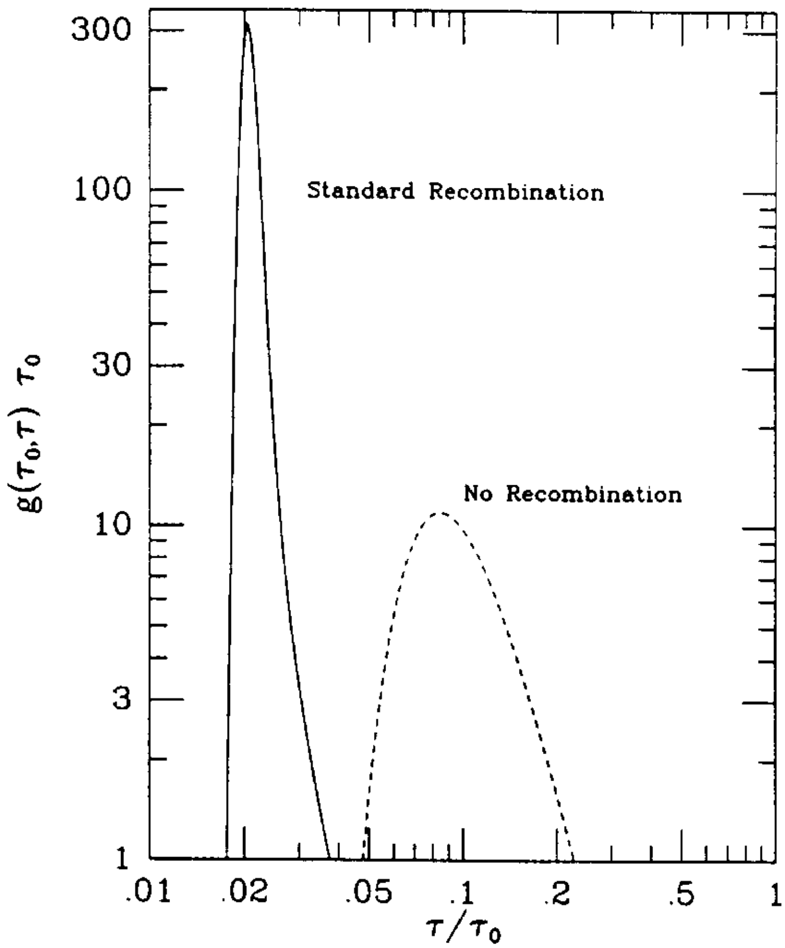

FIG. 7. The visibility functions for the ionization histories we will discuss in this section. Shown for comparison are the standard recombination history and the fully ionized history. The small numbers alongside each line indicate the redshift of complete ionization.

Figure 7 shows the visibility functions for the histories we will discuss. Note that the ionization in these particular histories is gradual so that even the history wherein ionization takes place at $z = 10$ differs from the standard recombination. The advantage of these particular histories is that they were generated in a consistent manner; i.e. a source of photons was injected continuously into the medium and the effects of recombination and electron heating were taken into account (Dodelson & Jubas 1992). The ionization history of the Universe *could* have been as sketched in any of the lines in Figure 7. The disadvantage of these histories is that in all of them re-ionization is gradual. We would like to be able to say something general about re-ionization without reference to these specific histories. Therefore, referring to them by the epoch of complete re-ionization is *not* a good idea.

It is more useful to discuss the *cumulative visibility function*: $\int_\tau^{\tau_0} d\tau' g(\tau_0, \tau')$; this is shown in Figure 8. The cumulative visibility function is the probability that a photon has last scattered after a given time. Thus, for a no-recombination Universe, Figure 8 shows that almost all photons scattered after $\tau = .05\tau_0$. By contrast, the cumulative visibility function is only equal to .005 at $\tau = .05\tau_0$ for the standard recombination history. The other ionization histories lie somewhere in between. A nice feature of this number [the cumulative visibility function at $\tau = .05\tau_0$] is that it characterizes each ionization history in an easily understandable way. Further, it lies between 0 [standard recombination] and 1 [no recombination] for all reasonable ionization histories.

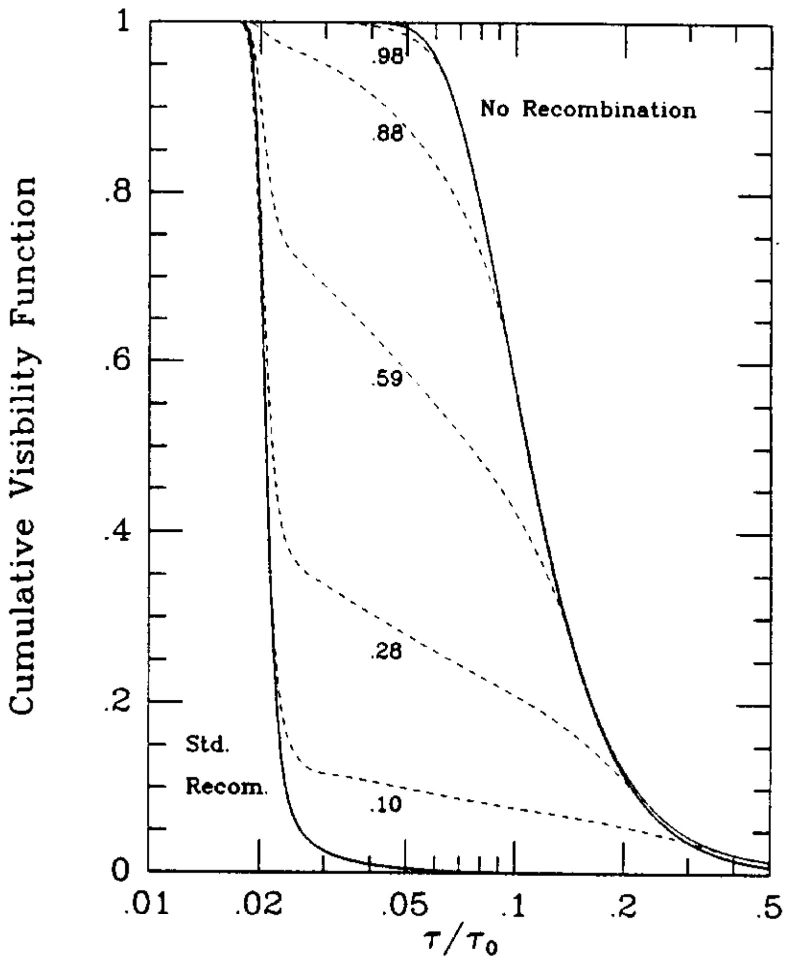

FIG. 8. The cumulative visibility functions for the ionization histories we will discuss in this section. The numbers by each denote the value of the cumulative visibility function when $\tau = 0.05\tau_0$.

For each of these histories we have calculated the expected signal in the CMB in the form of the C_l's. Figure 9 shows the results of the numerical integration of the Boltzmann equations. Working our way down from standard recombination, we see that if the cumulative visibility function is small at $\tau = .05\tau_0$ [e.g. the curves labeled .10,.28, and .59], the effect of reionization is to damp out the primary anisotropies generated early on. The more effective the reionization [as parametrized by a larger cumulative visibility function], the more significant is the damping of the peak at $l = 200$. Indeed, if most photons scattered late [$\tau > .05\tau_0$], then this peak goes away completely [e.g. the curves labeled .88 and .98], as we expect from our discussion in section 4 of the no recombination case. At the same

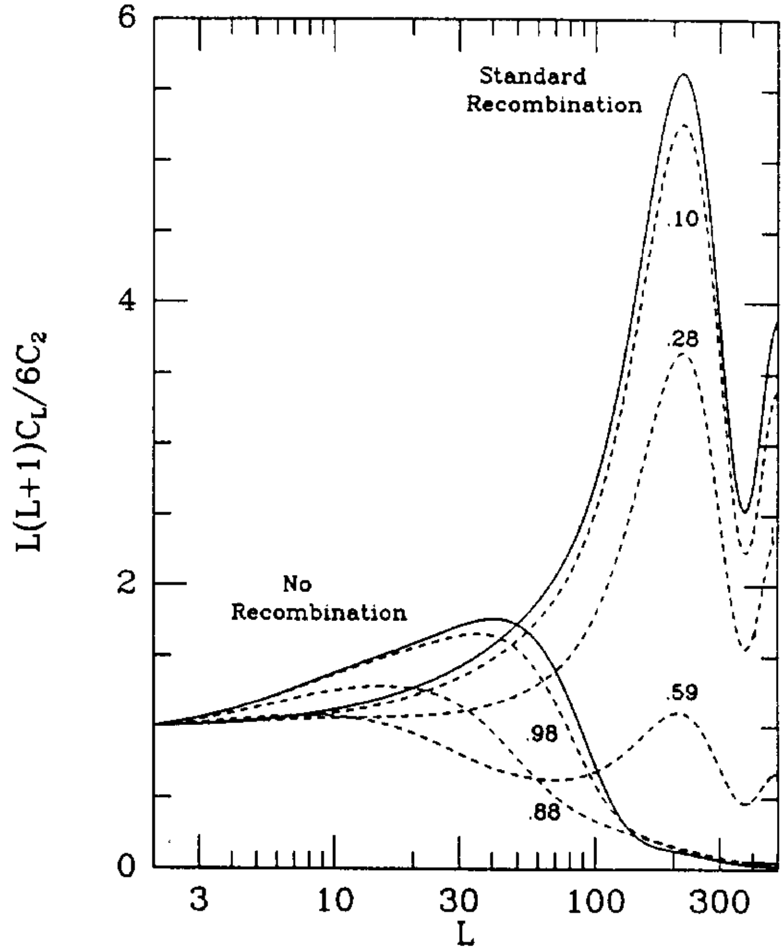

FIG. 9. The C_l's for different ionization histories ranging from standard recombination to no recombination. Again the numbers give the value of the cumulative visibility function at $\tau = .05\tau_0$.

time, though, secondary anisotropies – those generated at late times when velocities were larger – become important as reionization becomes more efficient. So for $l < 50$, C_l is actually *larger* in a Universe which reionizes early.

We now convolve these C_l's with the filter functions for the experiments plotted in Figure 1. We are interested in the expected signal, as defined in Eq. (1.1). So we plot $\langle \Delta T^2_{\rm expt} \rangle^{1/2}$ as a function of the reionization parameter.

The expected signal in the Tenerife experiment is virtually independent of the ionization history. We see a small increase in the signal for no-recombination vs. standard recombination since the Tenerife filter samples part of the Doppler no-recombination peak. But the difference is very small, of order 5%, certainly too small to be meaningful at present.

The South Pole 91 filter is situated between the standard recombination peak at $l \sim 200$ and the no-recombination peak at $l \sim 50$. Since the former peak has a larger amplitude, the signal in SP91 is larger if the Universe had a standard recombination history than if it never recombined. It is interesting to note though that the curve is not monotonic: If the cumulative visibility function at $\tau = .05\tau_0$ lay between 0.5 and 1, the

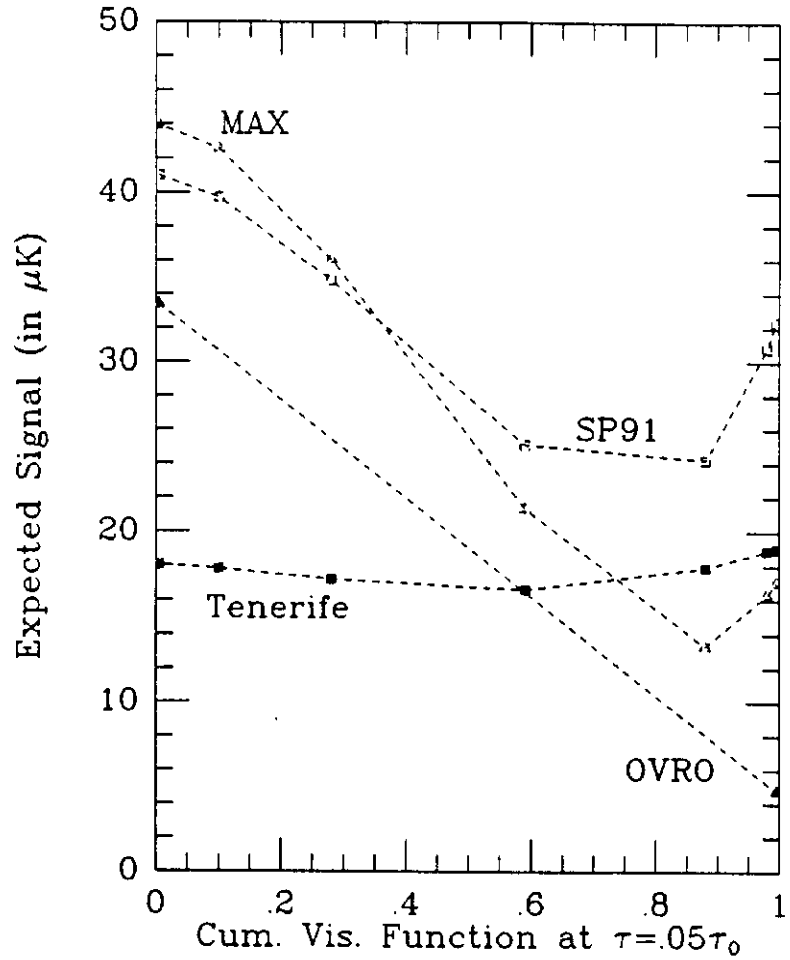

FIG. 10. The expected $< \Delta T^2 >^{1/2}$ in a variety of experiments as a function of the reionization parameter, the cumulative visibility function at $\tau = .05\tau_0$. Small values of the cumulative visibility function correspond to standard recombination; values close to one correspond to no recombination. The signal in OVRO includes the second order Vishniac effect.

Doppler peak at $l \sim 200$ is significantly depleted and the secondary peak at $l \sim 50$ has not built up to its maximum value. Thus the expected signal in medium scale anisotropy experiments is lowest if roughly half the photons last scattered at the standard $z \simeq 1100$ and the other half at $z \simeq 100$. The signal can drop by as much as a factor of two from what would be expected in standard recombination. Amusingly, the observed signal in all four channels *was* roughly a factor of two below what would be expected, but this must be taken with a grain of salt since there was probably a great deal of contamination from foreground sources (Gaier et al. 1992; Dodelson & Jubas 1993).

The MAX experiment has a filter centered closer to the primary peak at $l \simeq 200$. Therefore, the signal drops even more precipitously when the Universe is re-ionized and the primary peak is washed out. The signal can be a factor of three smaller than in standard recombination. Note again the effect of the secondary peak: as the cumulative visibility function reaches one, the secondary peak and, therefore, the signal increases. *MAX and*

SP91 are both very good probes of the ionization history, with the expected signal varying by a factor of $2-3$ *depending on the ionization history.*

By far the best probe of ionization history is the Owens Valley Radio Observatory receiver. Note from Fig. 1 that the OVRO filter does not pick up the secondary peak at $l \sim 50$. So the signal due to first order effects [pre-Vishniac] is completely negligible if the Universe never recombined. In fact, the expected signal drops by a factor of order ten if the Universe never recombined. This drop is large but it would be even more dramatic if not for the Vishniac effect, which produces a small, but non-negligible signal in the case of no-recombination when the primary signal has been completely washed out.

To sum up, Fig. 10 gives meat to the common wisdom that the smallest scale experiments are most sensitive to the ionization history of the Universe.

We started this investigation wondering whether a signal due to re-ionization could be misinterpreted as a primordial signal. We now know how a signal due to re-ionization differs from one due to standard CDM with standard recombination history, and we are confident that current and future experiments will probe such differences. What we have not done in this paper is explore how a signal due to re-ionization differs from a signal in other variants of CDM. Most troublesome are two variants most likely to be confused with re-ionization: (i) models where the primordial spectrum is *not* Harrison-Zel'dovich ($n < 1$) and (ii) models in which there are primordial tensor perturbations due to gravity waves (Crittenden *et al.* 1993). Both of these have the feature that the signal on large scales is the same as in standard CDM, while the signal on small scales is smaller than the standard one. They share this feature with re-ionized models. Therefore, distinguishing tilted or gravity-wave models from re-ionized models is a challenging task for cosmologists.

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